Notes for Étale Cohomology Seminar 1

藏雪

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The textbook we use is *Lectures on Etale Cohomology* (v2.21) by Milne, which should be available at https://www.jmilne.org/math/CourseNotes/lec.html.

1 Introduction

Recall the cohomology in algebraic topology: for a topological space X we have $H^*(X)$; for closed subset $Z \subset X$ we have $H^*_Z(X)$, and long exact sequence $\cdots \to H^{r-1}(U) \to H^r_Z(Y) \to H^r(X) \to H^r(U) \to \cdots$, where $U = Z - X$. This cohomology

- only depends on the homotopy class,
- has excision, and
- satisfies $H^r(\mathrm{pt}) = 0$.

We will construct another cohomology for algebraic objects, i.e. the étale cohomology, $H^r_{\text{ét}}(X, \Lambda)$. We will finally prove

Theorem 1 (Comparison Theorem)**.** *Suppose Λ is a finite abelian group, $X/\mathbb{C}$ is a nonsingular variety, then $H^r_{\text{ét}}(X, \Lambda) \cong H^r(X(\mathbb{C}), \Lambda)$.*

An application of étale cohomology is Weil conjectures. Suppose X is a projective smooth variety of dimension d over the finite field $\mathbb{F}_q$. Define $N_m := \#X(\mathbb{F}_{q^m})$, $Z(t) := \exp(\sum_{m\geq 1} N_m \cdot \frac{t^m}{m}) \in \mathbb{Q}[[t]]$, and define the zeta function $\zeta_X(s) := \prod_{x\in X} \frac{1}{1-(\#k(x))^{-s}}$.

Theorem 2 (Weil Conjectures)**.** *(a) $Z(t) \in \mathbb{Q}(t)$, and we have the functional equation*

$$Z(q^{-d}t^{-1}) = \pm(q^{d/2}t)^{\chi} Z(t),$$

where $\chi = (\Delta.\Delta)$ is the Euler-Poincare characteristic of X.

(b)

$$Z(t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)},$$

where $P_0(t) = 1 - t, P_{2d}(t) = 1 - q^{2d}t$, and for $1 \leq r \leq 2d-1$, $P_r(t) = \prod_{i=1}^{\beta_r}(1 - \alpha_{r,i}t)$, where $\alpha_{r,i} \in \mathbb{C}$, $|\alpha_{r,i}| = q^{r/2}$. The part "$|\alpha_{r,i}| = q^{r/2}$" is called the "Riemann hypothesis".

(c) Let β_r be the Betti numbers of X, then $\chi = \sum(-1)^r\beta_r$.

We introduce a heuristic proof here.

At first, we may prove the Lefschetz fixed point formula: If $\varphi : X \to X$, then

$$(\Gamma_\varphi.\Delta) = \sum_r (-1)^r \operatorname{Tr}(\varphi^* \mid H^r(X, ?)),$$

where "?" represents some suitable field of characteristic 0.

Take $\varphi = \mathrm{id}$, so we get $\chi = \sum_r (-1)^r \dim H^r(X, ?)$. Recall that $\beta_r = \dim H^r(X, ?)$, so (c) follows.

Let $F : X \to X$ be the Frobenius map (i.e. on the level of rings it is $a \mapsto a^q$). Note that $\operatorname{tr.d.} K(X)/\mathbb{F}_q = d$, so $[K(X) : K(X)^q] = [\mathbb{F}_q(t_1, \cdots, t_d) : \mathbb{F}_q(t_1^q, \cdots, t_d^q)] = q^d$, hence $\deg F = q^d$.

We can define the cup product $\smile$: $H^r(X) \to H^{2d-r} \to H^{2d}$, which is bilinear. For $\varphi : X \to X$, $\alpha \in H^{2d-r}(X), \beta \in H^r(X)$, we can show the formula

$$\varphi^*(\varphi_*\alpha \smile \beta) = \alpha \smile \varphi^*\beta,$$

thus $F_*F^* = \deg F = q^d$.

The fixed points of $F^m : X \to X$ are exactly $X(\mathbb{F}_{q^m})$, so let's admit the fact that $\#X(\mathbb{F}_{q^m}) = (\Gamma_{F^m}.\Delta)$. Thus $N_m = \sum_r (-1)^r \operatorname{Tr}((F^*)^m \mid H^r)$ by Lefschetz fixed point formula. Define $P_r(t) = \det(I - tF^* \mid H)$. Let $\alpha_{r,1}, \cdots, \alpha_{r,\beta_r} \neq 0$ be the eigenvalues of F^*, so $P_r(t) = \prod_{i=1}^{\beta_r}(1 - \alpha_{r,i}t)$.

Then we get

$$\sum_{m\geq 1} \operatorname{Tr}((F^*)^m \mid H^r)\frac{t^m}{m} = \sum_{m\geq 1}\sum_{i=1}^{\beta_r} \frac{\alpha_{r,i}^m t^m}{m} = \sum_{i=1}^{\beta_r} -\log(1 - \alpha_{r,i}t) = -\log P_r(t).$$

Taking the sum, we get

$$\log Z(t) = \sum_{m\geq 1} N_m \frac{t^m}{m} = \sum_r (-1)^{r-1} \log P_r(t),$$

hence the equation in (b). In particular, $P_0(t) = 1 - t$, $P_{2d}(t) = 1 - (\deg F)t = 1 - q^d t$.

Note that $F_{*,2d-r}$ is the dual of $F^{*,r}$, so their eigenvalues are the same. But we have shown $F_*F^* = q^d$, so the eigenvalues of $F^{*,2d-r}$ are $q^d/\alpha_{r,i}$. Thus the factors in $Z(t)$ also appear in pairs. To be precise, $Z(t) = \prod((1 - \alpha t)(1 - \frac{q^d}{a}t))^{\pm 1} \cdot \prod(1 \pm q^{d/2}t)^{\pm 1}$. Then check on each of the factors and we will obtain the desired functional equation in (a).

The "Riemann hypothesis" is much harder, so let's just skip it.

2 Étale morphisms

Most of the results are displayed without proofs. The reader who wants proofs should refer to the textbook or the Stacks Project.

Definition 3. $f : X \to Y$ is **unramified** if f is locally of finite presentation, and for all $x \in X, y = f(x)$ we have

(1) $\mathfrak{m}_{X,x} = \mathfrak{m}_{Y,y}\mathscr{O}_{X,x}$.

(2) $k(x)/k(y)$ is finite separable.

Theorem 4. *Suppose f is locally of finite presentation. TFAE:*

(1) f is unramified.

(2) The diagonal map is an open immersion.

(3) $\Omega_{X/Y} = 0$.

Definition 5. $f : X \to Y$ is **étale** if f is locally of finite presentation, flat, and $\forall y \in Y$, $f^{-1}(y)$ is a disjoint union of $\operatorname{Spec} k_i$, where each $k_i/k(y)$ is finite separable.

Theorem 6. *TFAE:*

(1) $f : X \to Y$ is étale.

(2) f is locally of finite presentation, flat, and for all $y \in Y$, let $\overline{k} = \overline{k(y)}$, then $(X_y)_{\overline{k}}$ is a disjoint union of copies of $\operatorname{Spec} \overline{k}$.

(3) $f : X \to Y$ is locally of finite presentation, flat, and $\Omega_{X/Y} = 0$.

(4) f is flat and unramified.

Theorem 7. *Suppose A is a commutative ring, $B = A[x_1, \cdots, x_n]_g/(f_1, \cdots, f_n)$ (where $f_i, g \in A[x_1, \cdots, x_n]$). Then B/A is étale iff $\operatorname{Jac}(f_1, \cdots, f_n) := \frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$ is invertible in B.*

In particular, if $B = A[x]_g/(P(x))$, P is monic, P' is invertible in B, then B/A is étale. In this case we say it is ***standard étale****.*

Theorem 8. *Let $f : X \to Y$ be étale. Then $\forall x \in X, y = f(x)$, $\exists$ open neighbourhood $\operatorname{Spec} B \ni x$, $\operatorname{Spec} A \ni y$ such that $f(\operatorname{Spec} B) \subset \operatorname{Spec} A$, and that $A \to B$ is a standard étale morphism.*

Proof. Stacks Tag 00UE. □

Proposition 9 (Proposition 2.11 in textbook)**.** *(a) Open immersions are étale.*

(b) Étale morphisms are stable under base change.

(c) Étale morphisms are stable under composition.

(d) Suppose $X \to Y \to Z$ such that $X \to Z$ étale, $Y \to Z$ unramified, then $X \to Y$ is étale.

Proof. (d) By Cancellation Lemma. Note that the diagonals of unramified morphisms are étale. □

Proposition 10 (Proposition 2.12 in textbook)**.** *Suppose $f: X \to Y$ is étale, Y is Noetherian.*

(a) $\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{Y,f(x)}$.

(b) f is quasi-finite.

(c) f is open mapping.

(d) If Y is reduced then so is X.

(e) If Y is normal then so is X.

(f) If Y is regular then so is X.

Proposition 11 (Proposition 2.15 in textbook)**.** *$p: X \to Y$ is étale separated, $s: Y \to X$ is a section. Then s is an open immersion and a closed immersion.*

Corollary 12. *Let $p: X \to S$ be étale separated, $q: Y \to S$, and φ, φ' be S-morphisms. Then the locus where φ and φ' agree is open and closed.*

Proof. Consider the sections $(\mathrm{id}, \varphi), (\mathrm{id}, \varphi'): Y \to Y \times_S X$ of $Y \times_S X \to Y$. □

3 The local ring for the étale topology

Definition 13. A **geometric point** of X is a morphism $\overline{x}: \operatorname{Spec} k \to X$, where k is separably closed.

Definition 14. Let $\overline{x}$ be a geometric point of X. A **étale neighbourhood** of $\overline{x}$ is a tuple $(U, \varphi, \overline{u})$, where $\overline{u}$ is a geometric point of U over $\overline{x}$, and $\varphi: U \to X$ is etale.

A **morphism** of étale neighbourhoods $f: (U, \varphi, \overline{u}) \to (V, \phi, \overline{v})$ is a morphism $f: U \to V$ (which is automatically étale), such that $\phi \circ f = \varphi$ and $f(\overline{u}) = \overline{v}$.

By Corollary 12, there is at most one morphism from a connected étale neighbourhood to another étale neighbourhood. Thus all the connected étale neighbourhoods form a direct set by $(U, \overline{u}) \leq (V, \overline{v})$ if there exists an étale morphism $(V, \overline{v}) \to (U, \overline{u})$.

Definition 15. Define the **local ring at $\overline{x}$ for the étale topology**,

$$\mathscr{O}_{X,\overline{x}} := \operatorname{colim}_{(U,\overline{u})} \Gamma(U, \mathscr{O}_U).$$

Proposition 16. *Suppose X is Noetherian. Then $\mathscr{O}_{X,\overline{x}}$ is a local Noetherian ring with Krull dimension* $= \dim \mathscr{O}_{X,x}$.

Theorem 17. $\mathscr{O}_{\mathbb{A}^n_k, \overline{0}} = \{f \in k^{\mathrm{sep}}[[x_1, \cdots, x_n]] \mid f$ *integral over* $k^{\mathrm{sep}}(x_1, \cdots, x_n)\}$.

Definition 18. Suppose A is local ring. Call A **Henselian** if $\forall f \in A[x]$ monic, $\overline{f} = \overline{g}\overline{h}$ in $(A/\mathfrak{m})[x]$, $\gcd(\overline{g}, \overline{h}) = 1$, $\exists$ liftings g, h of $\overline{g}, \overline{h}$ monic s.t. $f = gh$. Call A **strictly Henselian** if A is Henselian and $A/\mathfrak{m}$ is separably closed.

Theorem 19. *$\mathscr{O}_{X,\overline{x}}$ is the smallest strictly Henselian ring over $\mathscr{O}_{X,x}$, i.e. $\forall A$ strictly Henselian, $\mathscr{O}_{X,x} \hookrightarrow A$, $\exists$ unique $\mathscr{O}_{X,\overline{x}} \to A$ such that*

$$\begin{array}{ccc} \mathscr{O}_{X,\overline{x}} & & \\ \uparrow & \searrow & \\ \mathscr{O}_{X,x} & \longrightarrow & A \end{array}$$

commutes.

4 Sites

Definition 20. A **site** is a (small) category $\mathcal{C}$, together with a collection $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i \in I}$, called the **coverings**, such that

(1) $\forall$ isomorphism $V \to U$, $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.

(2) If $\{U_i \to U\}_i \in \mathrm{Cov}(\mathcal{C})$, and $\{V_{ij} \to U_i\}_j \in \mathrm{Cov}(\mathcal{C})$ for all i, then $(V_{ij} \to U)_{ij} \in \mathrm{Cov}(\mathcal{C})$.

(3) If $\{U_i \to U\}_i \in \mathrm{Cov}(\mathcal{C})$, and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all i, and $\{U_i \times_U V \to V\} \in \mathrm{Cov}(\mathcal{C})$.

Example. X is a topological space, $\mathcal{C} = \{\text{open sets of } X\}$, morphisms are inclusions, and coverings are interpreted normally, then it is a site.

Example. Suppose X is a scheme. The **small étale site** $X_{\text{ét}}$ is defined as follows: the objects are étale morphisms $U \to X$, morphisms are morphisms $U \to V$ with
$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow & \downarrow \\ & & X \end{array}$$
commutative, coverings are $(f_i : U_i \to U)$ such that $U = \bigcup f_i(U_i)$.

Tthe **big étale site** $X_{\text{Ét}}$ is defined as follows: the category is Sch/X, coverings are $(f_i : U_i \to U)$ such that f_i étale, and $U = \bigcup f_i(U_i)$.

Definition 21. A functor $\mathcal{F} : \mathcal{C}^{\text{op}} \to \mathsf{Set}$ is called a **presheaf**.

If for all coverings $(U_i \to U)_i$ we have an equalizer $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \mathcal{F}(U_i \times_U U_j)$, then $\mathcal{F}$ is called a **sheaf**.

Proposition 22. *Let $\mathscr{F}$ be a presheaf on $X_{\text{ét}}$. Then $\mathscr{F}$ is a sheaf $\iff$ $\mathscr{F}$ is a Zariski sheaf (i.e. is a sheaf in the Zariski sense, i.e. the equalizer condition holds for any covering $(U_i \to U)_i$ such that each $U_i \to U$ is open immersion) and $\forall V \to U$ surjective étale with U, V both affine, we have an equalizer $\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$.*

Example (Constant sheaf). Let Λ be a set, then $\mathcal{F}_\Lambda(U) := \Gamma(U, \underline{\Lambda})$ is an étale sheaf.

Proof. By definition, $\mathcal{F}_\Lambda$ is a Zariski sheaf.

Note that if Λ has the discrete topology, then $\Gamma(U, \underline{\Lambda}) = \{\text{continuous maps } U \to \Lambda\}$. Let $f : V \to U$ be étale surjective, then $\mathcal{F}_\Lambda(U) \to \mathcal{F}_\Lambda(V)$ is injective.

Let $\mathrm{pr}_1, \mathrm{pr}_2 : V \times_U V \to V$. Suppose $s \in \mathcal{F}_\Lambda(V)$ such that $\mathrm{pr}_1^* s = \mathrm{pr}_1^* s : V \times_U V \to \Lambda$. For any $u \in U$, $v_1, v_2 \in f^{-1}(u)$, we may take $w \in V \times_U V$ such that $\mathrm{pr}_1(w) = v_1$, $\mathrm{pr}_2(w) = v_2$. Thus $s(v_1) = (\mathrm{pr}_1^* s)(w) = (\mathrm{pr}_2^* s)(w) = s(v_2)$, hence s descends to a map $s' : U \to \Lambda$. Also $V \to U$ is étale, hence open, hence s' is continuous. Therefore $\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$ is exact. □

Example. Suppose Y is an X-scheme. Then $h_Y(U) := \mathrm{Hom}_{\mathsf{Sch}/X}(U, Y)$ is an étale sheaf. Take $Y = \mathbb{G}_a, \mathbb{G}_m, \mathrm{GL}_n$, respectively, so we get the corresponding étale sheaves.

Example. Suppose $\mathscr{F}$ is a quasi-coherent sheaf on X. Then $\mathscr{F}^{\text{ét}}(U) := \Gamma(U, \varphi^*\mathscr{F})$ (where $\varphi : U \to X$) is an étale sheaf.

Example (Skyscraper sheaf). Suppose $\overline{x}$ is a geometric point of X, Λ is a set. Then $\Lambda^{\overline{x}}(U) := \prod_{u \in \mathrm{Hom}_X(\overline{x}, U)} \Lambda$ is an étale sheaf.

Definition 23. Suppose $\mathcal{F}$ is a sheaf. Then the **stalk** $\mathcal{F}_{\overline{x}} := \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U)$.

Definition 24. $\mathcal{F}$ is called a **locally constant sheaf** if $\exists$ étale cover $(U_i \to X)$ such that $\mathcal{F}|_{U_i}$ are constant sheaves.