Notes for Étale Cohomology Seminar 2

藏雪

April 26, 2024

1 Étale fundamental group

Definition 1. Suppose X is locally Noetherian, $\overline{x} \in X$ is a geometric point.

The category $\mathsf{F\acute{E}t}/(X, \overline{x})$ is defined as follows: the objects are finite étale morphisms $(Y, \overline{y}) \to (X, \overline{x})$ such that $\overline{y} \mapsto \overline{x}$, and the map $\overline{y} = \operatorname{Spec} \Omega_1 \to \operatorname{Spec} \Omega = \overline{x}$ is an isomorphism. The category $\mathsf{F\acute{E}t}^\circ/(X, \overline{x})$ is the full subcategory of $\mathsf{F\acute{E}t}/(X, \overline{x})$ consisting of the connected schemes.

Similarly we define $\mathsf{F\acute{E}t}/X$ and $\mathsf{F\acute{E}t}^\circ/X$, where the objects are finite étale morphisms $Y \to X$ (in the latter one Y must be connected).

Recall that in topology, π_1^{top} can be given by the automorphism of the universal covering. Now we use a similar method to define $\pi_1^{\text{ét}}$. Generally, the "universal étale covering" does not exist, so we need to take the inverse limit on $\mathsf{F\acute{E}t}^\circ/(X, \overline{x})$.

Remark. Why do we use the category $\mathsf{F\acute{E}t}^\circ/(X, \overline{x})$ instead of $\mathsf{F\acute{E}t}/X$? The following proposition gives an answer: in this category, there is at most one morphism between any two objects, which makes it convenient to taking inverse limits.

Proposition 2. *Suppose* $(Y_1, \overline{y_1}), (Y_2, \overline{y_2}) \in \mathsf{F\acute{E}t}^\circ/(X, \overline{x})$.

(1) There is at most one morphism $(Y_1, \overline{y_1}) \to (Y_2, \overline{y_2})$.

(2) $\exists (Y, \overline{y})$ *such that both morphisms* $(Y, \overline{y}) \to (Y_i, \overline{y_i})$ *exist.*

Therefore, the objects in $\mathsf{F\acute{E}t}^\circ/(X, \overline{x})$ *form a (filtered) inverse set.*

Proof. (1) By the result from last week.

(2) Let Y be the connected component of $Y_1 \times_X Y_2$ that contains $(\overline{y_1}, \overline{y_2})$, and $\overline{y} = (\overline{y_1}, \overline{y_2})$. □

Proposition 3. *Suppose* $Y \in \mathsf{F\acute{E}t}^\circ/X$. *Then* $\# \operatorname{Aut}(Y/X) \leq \deg(Y/X)$, *where* $\operatorname{Aut}(Y/X)$ *is the group of isomorphisms of* Y *over* X.

Proof. We only prove when Y, X are integral. Then we have a natural map $\operatorname{Aut}(Y/X) \hookrightarrow \operatorname{Aut}(K(Y)/K(X))$. Thus we only need to show this map is injective. This can be checked on open subsets, so let's assume $X = \operatorname{Spec} A$; by finiteness, $Y = \operatorname{Spec} B$ is also affine. Now any map $B \to B$ is uniquely determined by the corresponding map $\operatorname{Frac}(B) \to \operatorname{Frac}(B)$, so we are done.

The general case is proved by reducing to integral case. See *Etale Cohomology Theory* by Fu Lei. ☐

Definition 4. A finite étale morphism $Y \to X$ is **Galois** if $\# \operatorname{Aut}(Y/X) = \deg(Y/X)$.

Proposition 5 (Proposition 3.2.10 in *Etale Cohomology Theory* by Fu Lei)**.** *Suppose* $(Y, \overline{y}) \in \mathsf{F\acute{E}t}^{\circ}/(X, \overline{x})$. *Then there is some* $(Z, \overline{z}) \in \mathsf{F\acute{E}t}^{\circ}/(X, \overline{x})$, *which is Galois, such that a morphism* $(Z, \overline{z}) \to (Y, \overline{y})$ *exists. Therefore, the Galois objects in* $\mathsf{F\acute{E}t}^{\circ}/(X, \overline{x})$ *also form a (filtered) inverse system.*

Proposition 6 (Mittag-Leffler property)**.** *Suppose* $Y', Y \in \mathsf{F\acute{E}t}^{\circ}/X$ *are Galois, such that a morphism* $Y' \to Y$ *exists. Then there is a canonical surjective map* $\operatorname{Aut}(Y'/X) \to \operatorname{Aut}(Y/X)$.

Proof. Reduce to integral case, then use the Galois theory of fields. ☐

Definition 7. With the two propositions above, we are able to define the **Étale fundamental group**

$$\pi_1^{\text{ét}}(X, \overline{x}) = \lim_{(Y, \overline{y}) \text{ Galois in } \mathsf{F\acute{E}t}^{\circ}/(X, \overline{x})} \operatorname{Aut}(Y/X).$$

Theorem 8. *Suppose* X *is connected,* $\overline{x_1}, \overline{x_2}$ *are geometric points. Then we have isomorphism* $\pi_1^{\text{ét}}(X, \overline{x_1}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{x_2})$, *which is canonical up to a conjugation.*

Theorem 9 (Comparison with π_1^{top})**.** *Suppose* X *is a variety (even not proper) over* $\mathbb{C}$, $x = \overline{x} \in X(\mathbb{C})$. *Then exists an equivalence of categories*

$$\begin{aligned} \mathsf{F\acute{E}t}/X &\overset{\sim}{\leftrightarrow} \{\textit{finite holomorphic covering of } X(\mathbb{C})\} \\ Y &\mapsto Y(\mathbb{C}) \end{aligned}$$

Therefore we have a canonical isomorphism $\pi_1^{\text{ét}}(X, \overline{x}) \xrightarrow{\sim} (\pi_1^{\text{top}}(X, x))\hat{}$.

Example. Suppose k is a field, and $\overline{x}$ be the geometric point with residue field k^{sep}. Then any object in $\mathsf{F\acute{E}t}^{\circ}/(\operatorname{Spec} k, \overline{x})$ is a finite separable extension of k. Therefore $\pi_1^{\text{ét}}(k, \overline{x}) = \operatorname{Gal}(k^{\text{sep}}/k)$.

Example. Suppose k is algebraically closed, $\operatorname{char} k = 0$. Take $\overline{x} = x = 1 \in \mathbb{G}_m$. Suppose $(Y, \overline{y}) \in \mathsf{F\acute{E}t}^{\circ}/(\mathbb{G}_m, \overline{x})$. Then Y is also a smooth affine curve. Let C be the complete curve containing Y, so we get a rational map $C \dashrightarrow \mathbb{P}^1$, which extends to a morphism $p : C \to \mathbb{P}^1$. Since Y is étale over $\mathbb{G}_m$, the only possible ramified points of p in $\mathbb{P}^1$ are 0 and ∞.

Let $e_1, \cdots, e_r$ and $f_1, \cdots, f_s$ be the ramification indexes of the preimages of 0 and ∞, respectively. By Riemann-Hurwitz formula ([Stacks] tag `0C1B`), we get $2g_C - 2 = -2\deg(p) +$

$\sum(e_i-1)+\sum(f_i-1)=-2\deg(p)+\sum e_i+\sum f_i-r-s=-r-s$. But $r\geq 1, s\geq 1, g_C\geq 0$, so $r=s=1, g_C=0$, hence $C\cong\mathbb{P}^1$, and $p^{-1}(0)$ and $p^{-1}(\infty)$ are single points.

By an automorphism on C, we may assume $p^{-1}(0)=\{0\}$, $p^{-1}(\infty)=\{\infty\}$. Then $Y=p^{-1}(\mathbb{G}_m)=C\backslash\{0,\infty\}=\mathbb{G}_m$, hence the morphism $Y\to X$ corresponds to some map $f:k[t,t^{-1}]\to k[t,t^{-1}]$. Since $f(t)$ is invertible, let $f(t)=at^n$, $a\in k\backslash\{0\}$, $n\in\mathbb{Z}$. By étale, $n\neq 0$; then we can assume $n>0$ and $a=1$ (otherwise apply an automorphism on $\mathbb{G}_m$). Therefore it is of the form $f_n:\mathbb{G}_m\to\mathbb{G}_m:t\mapsto t^n$. Clearly $\mathrm{Aut}(f_n)=\mu_n(k)$, hence $\pi_1^{\text{ét}}(\mathbb{G}_m,\overline{x})\cong\hat{\mathbb{Z}}$.

If $k=\mathbb{C}$, then $\mathbb{G}_m=\mathbb{C}\backslash\{0\}$, so $\pi_1^{\text{top}}(\mathbb{G}_m,x)\cong\mathbb{Z}$. Therefore, the previous theorem holds in this case.

Example. Suppose $X=\operatorname{Spec}\mathbb{Z}_p$, and x is the closed point. By the results introduced in last week, any element in $\mathsf{FÉt}^{\circ}/(\operatorname{Spec}\mathbb{Z}_p,\overline{x})$ is of the form $\operatorname{Spec}O_K\to\operatorname{Spec}\mathbb{Z}_p$, with $pO_K=\mathfrak{m}_{O_K}$ (use the definition of unramified morphisms). In the Galois case we can show $\mathrm{Aut}(\operatorname{Spec}O_K/\operatorname{Spec}\mathbb{Z}_p)\cong\mathrm{Aut}((\operatorname{Spec}O_K/\mathfrak{m}_{O_K})/\mathbb{F}_p)$. Hence $\pi_1^{\text{ét}}(\mathbb{Z}_p)\cong\pi_1^{\text{ét}}(\mathbb{F}_p)=\mathrm{Gal}(\mathbb{F}_p)\cong\hat{\mathbb{Z}}$.

Now consider $X\backslash\{x\}=\operatorname{Spec}\mathbb{Q}_p$. We have $\pi_1^{\text{ét}}(\mathbb{Q}_p)=\mathrm{Gal}(\mathbb{Q}_p)$, and an exact sequence $1\to I_p\to\mathrm{Gal}(\mathbb{Q}_p)\to\mathrm{Gal}(\mathbb{F}_p)\to 0$. We say I_p "records the ramification at the point x".

Proposition 10. $\pi_1^{\text{ét}}(\mathbb{P}_k^n)=\mathrm{Gal}(k)$.

Proposition 11. *$\pi_1^{\text{ét}}$ is a covariant functor of pointed schemes. Suppose we have a morphism $(X,\overline{x})\to(Y,\overline{y})$ such that $\Omega_{\overline{x}}=\Omega_{\overline{y}}$. Then there exists a canonical map $\pi_1^{\text{ét}}(X,\overline{x})\to\pi_1^{\text{ét}}(Y,\overline{y})$.*

Proposition 12. *Suppose X/k is a variety that is geometric connected. Then there exists a short exact sequence $1\to\pi_1(X_{k^{\text{sep}}},\overline{x})\to\pi_1(X,\overline{x})\to\mathrm{Gal}(k)\to 1$, where $\overline{x}\in X(k^{\text{sep}})$.*

2 Sheaves on sites

In this section, we will give several examples of sheaves on $X_{\text{ét}}, X_{\text{fpqc}}$, etc. We use the following notations. Suppose T is a site. Then PSh_T [resp. Sh_T] is the category of presheaves [resp. sheaves] of sets on T; PAb_T [resp. Ab_T] is the category of presheaves [resp. sheaves] of abelian groups on T.

Definition 13. A morphism $f:Y\to X$ is **fpqc** if it is faithfully flat, and for any affine open $U\subset X$, there exists finitely many affine opens $V_i\subset f^{-1}(U)$ such that $\{f(V_i)\}$ cover U.

Theorem 14 (Grothendieck)**.** *Representable functor is a sheaf on the fpqc site. Thus it does on $X_{\text{ét}}, X_{\text{fppf}}$.*

Proof. Similar as the result in the last week, for a presheaf $\mathcal{F}$ on the fpqc site to be a sheaf, we only need to show that it is a Zariski sheaf, and for any $f : \operatorname{Spec} B \to \operatorname{Spec} A$ faithfully flat, we have equalizer $\mathcal{F}(\operatorname{Spec} A) \to \mathcal{F}(\operatorname{Spec} B) \rightrightarrows \mathcal{F}(\operatorname{Spec} B \otimes_A B)$ (see [Stacks] tag `0301`).

Let $Z \in \mathsf{Sch}/X$, and $k_Z = \operatorname{Hom}_{\mathsf{Sch}/X}(-, Z) \in \mathsf{PSh}_{\mathsf{Sch}/X}$. Clearly k_Z is a Zariski sheaf, since we can glue morphisms between schemes. Therefore we need to show that for any $f : A \to B$ faithfully flat, we have equalizer $\operatorname{Hom}_{\mathsf{Sch}/X}(\operatorname{Spec} A, Z) \to \operatorname{Hom}_{\mathsf{Sch}/X}(\operatorname{Spec} B, Z) \rightrightarrows \operatorname{Hom}_{\mathsf{Sch}/X}(\operatorname{Spec} B \otimes_A B, Z)$.

First we reduce to the case $X = \operatorname{Spec} \mathbb{Z}$. Now by dividing Z into affine open subsets U_i, we may assume $Z = \operatorname{Spec} C$ is affine. Therefore finally we reduce to showing $E_{A \to B} : 0 \to A \xrightarrow{g} B \to B \otimes_A B$ is exact. Clearly g is injective. It suffices to show that if $b \in B$ such that $1 \otimes b - b \otimes 1 = 0$, then $b \in g(A)$.

Case 1: Assume g admits a section, i.e. there exists $s : B \to A$ such that $s \circ g = \mathrm{id}$. Then there is a map $h : B \otimes_A B \to B : b_1 \otimes b_2 \mapsto b_1 g(s(b_2))$. Then $0 = h(1 \otimes b - b \otimes 1) = g(s(b)) - b$, so $b = g(s(b)) \in g(A)$.

Case 2: Suppose $A \to A'$ is faithfully flat, let $B' = B \otimes_A A'$. If $E_{A' \to B'}$ is exact, then so is $E_{A \to B}$, since $E_{A' \to B'} = E_{A \to B} \otimes_A A'$.

Case 3: General case. Take $A' = B$ in case 2. Then $B \to B \otimes_A B$ admits a section $s : B \otimes_A B \mapsto B : b_1 \otimes b_2 \mapsto b_1 b_2$. By case 1, $E_{A' \to B'}$ is exact, hence $E_{A \to B}$ is exact. □

Example. For $Z \in \mathsf{Sch}/X$, we write $k_Z = \operatorname{Hom}_{\mathsf{Sch}/X}(-, Z) \in \mathsf{Sh}_X$.

(1) We have the **structure sheaf** $k_{\mathbb{G}_a}(U) = \operatorname{Hom}_{\mathsf{Sch}/X}(U, \mathbb{G}_a) = \Gamma(U, \mathscr{O}_U)$, where $\mathbb{G}_a$ is $\mathbb{G}_{a,X} = X \times_{\mathbb{Z}} \mathbb{Z}[t]$. By abuse of notation, write $\mathbb{G}_a = k_{\mathbb{G}_a}$. Similarly we have sheaves $\mathbb{G}_m$, GL_n.

(2) For any coherent Zariski sheaf $\mathscr{M} \in \mathsf{Coh}_{X_{\mathrm{zar}}}$, define $\mathscr{M}^{\text{ét}}$ by $\mathscr{M}^{\text{ét}}(U) := \Gamma(U, (U \to X)^* \mathscr{M})$, for any étale morphism $U \to X$. Therefore $\mathscr{M}^{\text{ét}} \in \mathsf{PAb}_{X_{\text{ét}}}$. Similarly we can define $\mathscr{M}^{\mathrm{fpqc}}$, $\mathscr{M}^{\mathrm{fppf}}$, etc.

Exercise: show that $\mathscr{M}^{\text{ét}}$ is a sheaf. (Hint: similar to Theorem 14.)

Example. We will use some properties of $\mathsf{Ab}_{X_{\text{ét}}}$ that haven't been introduced yet. It's not hard to accept them as analogies with the theory of sheaves on a scheme.

(1) **Kummer sequence**. Let $\mu_n = X \times_{\mathbb{Z}} \mathbb{Z}[t]/(t^n - 1)$, so $k_{\mu_n}(U) = \{a \in \Gamma(U, \mathscr{O}_U) \mid a^n = 1\}$. Denote the sheaf k_{μ_n} by μ_n also. If $n \in \Gamma(X, \mathscr{O}_X)^*$, then we have an exact sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 0$$

in $\mathsf{Ab}_{X_{\text{ét}}}$. In fact, for any $U \in X_{\text{ét}}$, clearly $0 \to \mu_n(U) \to \mathbb{G}_m(U) \xrightarrow{(\cdot)^n} \mathbb{G}_m(U)$ is exact. Hence it suffices to show $(\cdot)^n : \mathbb{G}_m \to \mathbb{G}_m$ is surjective. For any $f \in \mathbb{G}_m(U)$, we need to show that there exists an étale covering $(V_i \to U)$ of U such that each $f\,|_{V_i}$ is an n-th power. For any $\operatorname{Spec} A \subset U$, consider $B = A[t]/(t^n - f)$. Since $n \in \Gamma(X, \mathscr{O}_X)^*$, the element $(t^n - f)' = nt^{n-1}$ is invertible in B, hence B is étale over A, as stated in last week. Gluing the different $\operatorname{Spec} B$'s, we can show that $V = \mathbf{Spec}\, \mathscr{O}_U[t]/(t^n - f)$ is étale surjective over U, hence $(V \to U)$ is an étale covering. Clearly $f\,|_V$ is an n-th power, as desired.

(2) **Artin-Schreier sequence**. Similarly, if X is a scheme over $\mathbb{F}_p$, then we get an exact sequence

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_a \xrightarrow{t \mapsto t^p - t} \mathbb{G}_a \to 0$$

in $\mathsf{Ab}_{X_{\text{ét}}}$. Here $\underline{\mathbb{Z}/p\mathbb{Z}}$ is the constant sheaf, or equivalently, is the sheaf $k_{X \times_{\mathbb{F}_p} \mathbb{F}_p[t]/(t^p - t)}$.