Notes for Étale Cohomology Seminar 3

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1 Morphism between sites

Definition 1 (Morphism between sites)**.** Suppose T_1, T_2 are sites, $\mathsf{Cat}(T_1), \mathsf{Cat}(T_2)$ are underlying categories. A **morphism** $f : T_1 \to T_2$ is a functor $F : \mathsf{Cat}(T_2) \to \mathsf{Cat}(T_1)$ that preserves coverings and fibred products.

Remark. The direction of the morphism f is opposite to direction of the functor F. This can be illustrated by the following examples.

Example. Suppose $f : X \to Y$ is a map between topological spaces, then the pullback $F : \mathsf{Open}(Y) \to \mathsf{Open}(X)$ gives a morphism between the corresponding sites from X to Y.

Example. Suppose $f : X \to Y$ is a morphism between schemes. Then $f^* : \mathsf{Cat}(Y_{\text{ét}}) \to \mathsf{Cat}(X_{\text{ét}}) : U \mapsto U \times_Y X$ gives a morphism $X_{\text{ét}} \to Y_{\text{ét}}$.

2 Sheafification

Generally, it's hard to define "points" in a site, and the stalks at points won't behave as well as those in topological spaces. Therefore we use a different approach to define sheafification, i.e. the analogue of Čech cohomology.

Suppose $\mathcal{F} \in \mathsf{PSh}_T$. For a covering $\{V_\alpha\}$ of U, let

$$\check{H}^0(\mathcal{F}, U, \{V_\alpha\}) := \ker(\prod_\alpha \mathcal{F}(V_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(V_\alpha \times_U V_\beta)).$$

Define

$$\mathcal{F}^+(U) := \operatorname{colim}_{\text{covering}\{V_\alpha\}} \check{H}^0(\mathcal{F}, U, \{V_\alpha\}).$$

Then $\mathcal{F}^+$ is a separated sheaf (i.e. for any covering $\{V_\alpha \to U\}$, the map $\mathcal{F}(U) \to \prod \mathcal{F}(V_i)$ is injective), and $\mathcal{F}^{++}$ is a sheaf. The proofs can be found in [Stacks] tag `00W1`.

Definition 2 (Sheafification)**.** $\mathcal{F}^\# := \mathcal{F}^{++}$ is called the **sheafification** of $\mathcal{F}$, or the **sheaf associated to** $\mathcal{F}$.

Clearly there are natural morphisms $\mathcal{F} \to \mathcal{F}^+$, and hence $\mathcal{F} \to \mathcal{F}^\#$.

Proposition 3. *Suppose $\mathcal{F} \in \mathsf{PSh}_T$.*

(1) $\mathrm{Hom}_{\mathsf{PSh}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathsf{Sh}}(\mathcal{F}^\#, \mathcal{G})$, $\forall \mathcal{G} \in \mathsf{Sh}$*;*

(2) $(\cdot)^\# : \mathsf{PAb} \to \mathsf{Ab}$ *is exact.*

(3) If the site admits the concept of "stalks", e.g. $X_{\text{ét}}$, then $\mathcal{F}_x = \mathcal{F}_x^+$. Hence $(\cdot)^\#$ preserves stalks.

Remark (Sections of sheafification). Suppose $\mathcal{C}$ is a site, $\mathcal{F} \in \mathsf{PSh}(\mathcal{C})$, $U \in \mathcal{C}$. Then the elements in $\mathcal{F}^\#(U)$ are $s = (s_i)$, where $s_i \in \mathcal{F}(U_i)$, $\{U_i \to U\}$ is a covering, such that $\forall i, j$, $\exists \{W_{ijk} \to U_i \times_U U_j\}$ such that $s_i|_{W_{ijk}} = s_j|_{W_{ijk}}$, $\forall k$. Two elements $s = t$ if there exists a refinement $(U_i \to U)$ of the corresponding coverings of s, t, with $s = (s_i), t = (t_i)$, such that $s_i = t_i, \forall i$.

Proposition 4. *Suppose $\mathcal{F}, \mathcal{G} \in \mathsf{Ab}(\mathcal{C})$, $\varphi : \mathcal{F} \to \mathcal{G}$, then φ is surjective (epimorphism) iff every section $s \in \mathcal{G}(U)$ can be locally given by sections in $\mathcal{F}$.*

2.1 Example of sheafification: Picard

Definition 5. Suppose $X \to S$ is a S-scheme. The **relative Picard functor** $\underline{\mathrm{Pic}}_{X/S} : \mathsf{Sch}_S \to \mathsf{Set} : Y \mapsto \mathrm{Pic}(X \times_S Y)/f_Y^* \mathrm{Pic}(Y)$, where $f_Y : X \times_S Y \to Y$.

Then $\underline{\mathrm{Pic}}_{X/S} \in \mathsf{PAb}_{\mathsf{Sch}_S}$. We have $\mathsf{Cat}(\mathsf{Sch}_{S,\mathrm{zar}}) \subset \mathsf{Cat}(\mathsf{Sch}_{S,\text{ét}}) \subset \mathsf{Cat}(\mathsf{Sch}_{S,\mathrm{fppf}}) \subset \mathsf{Sch}_S$ on the underlying categories (here all are big sites). We can take sheafification of $\underline{\mathrm{Pic}}_{X/S}$ on all of them. In particular we get a sequence $\underline{\mathrm{Pic}}_{X/S} \to \underline{\mathrm{Pic}}_{X/S,\mathrm{zar}} \to \underline{\mathrm{Pic}}_{X/S,\text{ét}} \to \underline{\mathrm{Pic}}_{X/S,\mathrm{fppf}}$, where the latter one is the sheafification of the former one.

Remark. In the previous example, we ignore the set-theoretic problems, so sheafifications can be defined on big sites. Strictly speaking, we should do this by putting together the sheafifications on corresponding small sites T_*, $T \in \mathsf{Sch}_S$, $* \in \{\mathrm{zar}, \text{ét}, \mathrm{fppf}\}$.

Example. Generally, the picard functor is not representable. Suppose $S = \operatorname{Spec} k$, $X = \mathbb{A}_k^1$. Then we claim that $\underline{\mathrm{Pic}}_{X/S}$ is not representable (in fact it's not representable in any of the above senses).

Suppose $\underline{\mathrm{Pic}}_{X/k}$ were representable by some scheme T. Let C be the (complete) cuspidal curve over k. Any morphism $C \to T$ is determined by the corresponding $\widetilde{C} \to T$, where $\widetilde{C}$ is the normalization of C, so $\mathrm{Hom}_k(C, T) \to \mathrm{Hom}_k(\widetilde{C}, T)$ is injective. Hence $\underline{\mathrm{Pic}}_{X/k}(C) \to \underline{\mathrm{Pic}}_{X/k}(\widetilde{C})$ would be injective; with $\underline{\mathrm{Pic}}_{X/k}(\widetilde{C}) = 0$, we get $\underline{\mathrm{Pic}}_{X/k}(C) = 0$. However, a theorem shows that $\underline{\mathrm{Pic}}_{C/k}$ is represented by $\mathbb{G}_a \times \mathbb{Z}$, so $\mathrm{Pic}(X \times C) = \mathrm{Pic}(X \times C)/\mathrm{Pic}(X) = \underline{\mathrm{Pic}}_{C/k}(X) = \mathrm{Hom}_k(X, \mathbb{G}_a \times \mathbb{Z}) = \mathrm{Hom}_k(X, \mathbb{G}_a) \times \mathbb{Z} = (k[t], +) \times \mathbb{Z}$, and $\mathrm{Pic}(C) = \mathbb{G}_a \times \mathbb{Z}$. Hence $\underline{\mathrm{Pic}}_{X/k}(C) = \mathrm{Pic}(X \times C)/\mathrm{Pic}(C) \neq 0$, a contradiction.

Remark. With a method similar to [Hartshorne] (II, Ex 7.13), we can show that $\text{Pic}(X \times C) = (k[t], +) \times \mathbb{Z}$ directly, without assuming the representability of $\underline{\text{Pic}}_{C/k}$.

Theorem 6. *Suppose X is projective over Z, S is Noetherian flat with geometrically integral fibers. Then $\underline{\text{Pic}}_{X/S,\text{ét}}$ is represented by a separated scheme locally of finite type over S.*

Example. Suppose $S = \operatorname{Spec} \mathbb{R}$, $X = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$. Then $\underline{\text{Pic}}_{X/S}$ is not representable, but by the previous theorem, $\underline{\text{Pic}}_{X/S,\text{ét}}$ is representable. Now we illustrate what happens during the sheafification.

Suppose $T = \operatorname{Spec} \mathbb{C}$, so $X_T = X \times_S T \cong \mathbb{P}^1_{\mathbb{C}}$. Recall that $\text{Pic}(X_T)/\text{Pic}(T) = \text{Pic}(\mathbb{P}^1_{\mathbb{C}})$ is the free abelian group generated by $\mathscr{O}(1)$. However, any closed point on X is $\mathbb{C}$-valued $\implies$ has degree 2 over $\mathbb{R}$ $\implies$ any invertible sheaf in $\text{Pic}(X)$ has an even degree $\implies$ $\text{Pic}(X)/\text{Pic}(S) = \text{Pic}(X) = 2\mathbb{Z} \cdot \mathscr{O}(1)$ (as a subset of $\text{Pic}(\mathbb{P}^1_{\mathbb{C}})$) $\implies$ $\mathscr{O}(1) \notin \underline{\text{Pic}}_{X/S}(S)$. On the other hand, $\{T\}$ is étale covering of S, and $\mathscr{O}(1) \in \underline{\text{Pic}}_{X/S}(T)$. Therefore $\mathscr{O}(1)$ is added to $\underline{\text{Pic}}_{X/S,\text{ét}}(S)$ during the sheafification.

3 Sheaves on the étale site

Recall that we have defined stalks on $X_{\text{ét}}$ previously, as $\mathcal{F}_{\overline{x}} := \operatorname{colim}_{(U,\overline{u})} \mathcal{F}(U)$, where $\overline{x}$ is any geometric point of X.

Proposition 7. *Suppose $\varphi : \mathcal{F} \to \mathcal{F}'$ is a morphism in $\mathsf{Ab}(X_{\text{ét}})$, and $\varphi_{\overline{x}} : \mathcal{F}_{\overline{x}} \to \mathcal{F}'_{\overline{x}}$ are maps on stalks. Then*

(1) φ is surjective iff all $\varphi_{\overline{x}}$ are surjective.

(2) φ is injective iff all $\varphi_{\overline{x}}$ are injective.

(3) φ is an isomorphism iff all $\varphi_{\overline{x}}$ are isomorphisms.

Proof. (1) Checking the universal properties, $\operatorname{coker} \varphi$ exists and equals to $(\mathcal{F}'/\varphi(\mathcal{F}))^{\#}$. Therefore $(\operatorname{coker} \varphi)_{\overline{x}} = ((\mathcal{F}'/\varphi(\mathcal{F}))^{\#})_{\overline{x}} = (\mathcal{F}'/\varphi(\mathcal{F}))_{\overline{x}} = \operatorname{coker} \varphi_{\overline{x}}$. Thus it suffices to show that if $\mathcal{G}_{\overline{x}} = 0, \forall \overline{x}$, then $\mathcal{G} = 0$. In fact, suppose $s \in \mathcal{G}(U)$. For any $\overline{x} \to U$, $0 = s_{\overline{x}} = \operatorname{colim}_{\overline{x} \to V \to U} s|_V$, hence $s|_V = 0$ for some V. All such V form a covering of U. By the sheaf axioms, $s = 0$.

(2) Similar. Note that the kernel of φ (as morphism between presheaves) is already a sheaf.

(3) By (1)(2), it suffices to show if φ is surjective and injective, then it is an isomorphism. First, by the construction of $\ker \varphi$ in (2), we get $\mathcal{F}(U) \subset \mathcal{F}'(U)$, $\forall U$. For any $s \in \mathcal{F}'(U)$, we get $s|_{\overline{x}} \in \mathcal{F}'_{\overline{x}} = \mathcal{F}_{\overline{x}}$. Each gives a section in some $\mathcal{F}(V_{\overline{x}})$ for some $V_{\overline{x}}$, so $s|_{V_{\overline{x}}} \in \mathcal{F}(V_{\overline{x}})$ (as element in $\mathcal{F}'(V_{\overline{x}})$). By the sheaf axioms, we get $s \in \mathcal{F}(U)$. Therefore $\mathcal{F}(U) = \mathcal{F}'(U)$, as desired. ∎

Theorem 8. $\mathsf{Ab}(X_{\text{ét}})$ *is an abelian category with limits and colimits. Filtered colimits are exact.*

Proof. We have proved the existence of kernels and cokernels. Since injectivity and surjectivity are checked on stalks, we can show that every injective morphism is a kernel of some morphism, and every surjective morphism is a cokernel of some morphism. Also, obviously arbitrary products and coproducts exist. Therefore it is an abelian category with limits and colimits.

Note that cokernels of sheaves are sheafifications of cokernels as presheaves, and coproducts as presheaves are already sheaves, so any colimit of sheaves is sheafification of colimit as presheaves. Clearly, in $\mathsf{PAb}(X_{\text{ét}})$, filtered colimits are exact; also sheafification is exact, so this is still true for $\mathsf{Ab}(X_{\text{ét}})$. $\square$

Theorem 9. $\mathsf{Ab}(X_{\text{ét}})$ *has enough injectives.*

Proof. $\mathsf{Ab}(X_{\text{ét}})$ has a generator $\mathcal{G} = \bigoplus_{V \to X \text{ étale}} \mathbb{Z}|_V$ (to be a **generator** means $\mathrm{Hom}(\mathcal{G}, -)$ is faithful). Thus $\mathsf{Ab}(X_{\text{ét}})$ is Grothendieck category, so it has enough injectives (see [Stacks] tag `079A`, `05AB`). $\square$

4 Pushforward, pullback

Definition 10. Suppose $\pi : Y \to X$ is a morphism of schemes.

(1) Suppose $\mathcal{F} \in \mathsf{PSh}(Y_{\text{ét}})$. The **pushforward** $\pi_*\mathcal{F}$ is a presheaf on $X_{\text{ét}}$, given by $\pi_*\mathcal{F}(U) := \mathcal{F}(U \times_X Y) \in \mathsf{PSh}(X_{\text{ét}})$. If $\mathcal{F}$ is a sheaf, then so is $\pi_*\mathcal{F}$.

(2) Suppose $\mathcal{F} \in \mathsf{Sh}(X_{\text{ét}})$. Then we get the presheaf $\mathcal{G} : U \mapsto \operatorname{colim} \mathcal{F}(V) \in \mathsf{PSh}(Y_{\text{ét}})$, where colimit is taken over all commutative diagrams (not necessarily cartesian)

$$\begin{array}{ccc} U & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \text{étale} \\ Y & \longrightarrow & X \end{array}$$

The **pullback** $\pi^*\mathcal{F} := \mathcal{G}^\#$, which is a sheaf on $Y_{\text{ét}}$.

In (2), if $\overline{y}, \overline{x}$ are geometric points on Y, X, respectively, such that $\overline{y} \mapsto \overline{x}$, then $(\pi^*\mathcal{F})_{\overline{y}} = \mathcal{G}^\#_{\overline{y}} = \mathcal{G}_{\overline{y}} = \mathcal{F}_{\overline{x}}$. Therefore

Proposition 11. π^* *preserves stalks. In particular,* π^* *is exact (as functor between* Ab*).*

By definition, for $\mathcal{H} \in \mathsf{PSh}(Y_{\text{ét}})$, we get $\mathrm{Hom}_{\mathsf{PSh}(Y_{\text{ét}})}(\mathcal{G}, \mathcal{H}) = \{$collection of $\begin{array}{ccc} U & \to & V \\ \downarrow & & \downarrow\text{étale} \\ Y & \to & X \end{array}$ $\mapsto (\mathcal{F}(V) \mapsto \mathcal{H}(U))\} = \mathrm{Hom}_{\mathsf{PSh}(X_{\text{ét}})}(\mathcal{F}, \pi_*\mathcal{H})$. Hence

Proposition 12. (π^*, π_*) *are adjoint pairs. As a corollary,* π_* *preserves injective objects.*

Remark. Generally, if the left adjoint is exact, then the right adjoint preserves injective objects (try it yourself).

Definition 13. Suppose $j : U \to X$ is an open immersion. For $\mathcal{F} \in \mathsf{Ab}(U_{\text{ét}})$, let

$$\mathcal{F}_! \in \mathsf{PAb}(X_{\text{ét}}), V \mapsto \begin{cases} \mathcal{F}(V), & V \to X \text{ factors through } U, \\ 0, & \text{otherwise.} \end{cases}$$

Define the **extension by zero** $j_!\mathcal{F} = (\mathcal{F}_!)^{\#} \in \mathsf{Ab}(X_{\text{ét}})$.

By definition, if $\overline{x} \to X$ factors through U, then $(j_!\mathcal{F})_{\overline{x}} = (j_*\mathcal{F})_{\overline{x}} = \mathcal{F}_{\overline{x}}$. If $\overline{x} \to X$ does not factor through U, then $(j_!\mathcal{F})_{\overline{x}} = 0$, but it's hard to calculate $(j_*\mathcal{F})_{\overline{x}}$.

Similarly we can show that

Proposition 14. *(1) $(j_!, j^*)$ are adjoint pairs.*

(2) $j_!$ is exact.

(3) j^* preserves injectives.

4.1 Finite pushforward

We give a formula for calculating stalks of finite pushforwards. An important application is the exactness of finite pushforward.

Lemma 15. *Suppose X is a scheme, $\{Y_i\}$ is a cofiltered inverse system of X-schemes, and $Y_j \to Y_i$ are affine. Then $Y = \lim Y_i$ exists (show affine locally). Suppose Y_i are quasi-compact and quasi-separated as schemes, and $Z \to X$ is locally of finite presentation. Then the map* $\operatorname{colim} \operatorname{Hom}_X(Y_i, Z) \to \operatorname{Hom}_X(Y, Z)$ *is an isomorphism.*

Proof. Affine case: By definition of direct limits of A-algebras. Need to use Z is of finite presentation.

General case: Fix some Y_0 in the inverse system. Take affine cover $\{U_i\}$ of X. Take affine cover $\{V_{ij}\}$ of Z such that each V_{ij} maps into U_i. Take affine cover $\{W_{ijk}\}$ of Y_0 such that each W_{ijk} maps into V_{ij}. Then use quasi-compactness and quasi-separatedness to glue. □

Theorem 16. *Suppose $\pi : Y \to X$ is quasi-compact and quasi-separated, $\overline{x} \to X$ is a geometric point. Let $\widetilde{X} = \operatorname{Spec} \mathscr{O}_{X,\overline{x}}$, and $\widetilde{Y} = \widetilde{X} \times_X Y$, so we get*

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \widetilde{X} & \xrightarrow{f} & X \end{array}$$

Suppose $\mathcal{F} \in \mathsf{Ab}(Y_{\text{ét}})$. Then $(\pi_*\mathcal{F})_{\overline{x}} \to \operatorname{colim}_{\widetilde{Y} \to U' \to Y} \mathcal{F}(U')$ is an isomorphism.

Remark. Note that the right side is just the unsheafified version $\Gamma(\widetilde{Y}, (f')^*_{\text{pre}}\mathcal{F})$. In fact this is still true when the right side is replaced by $\Gamma(\widetilde{Y}, (f')^*\mathcal{F})$: See [Stacks] tag `03Q9`. Also, recall that the local ring for the étale topology $\mathscr{O}_{X,\overline{x}} := \operatorname{colim}_{(U,\overline{u}):\ U \text{ étale},\ \overline{u}\mapsto\overline{x}} \Gamma(U, \mathscr{O}_U)$.

Proof. By definition we need to show $\operatorname{colim}_{\overline{x}\to U\to X,\ U \text{ affine}} \mathcal{F}(Y\times_X U) \to \operatorname{colim}_{\widetilde{Y}\to U'\to Y} \mathcal{F}(U')$ is isomorphism. It suffices to prove $\widetilde{Y} \to Y\times_X U \to Y$ form a cofinal system in $\{\widetilde{Y} \to U' \to Y\}$ (in a direct system indexed by a partially ordered set A, a subset $B \subset A$ is **cofinal** if for any $a \in A$, there is some $b \in B$ such that $a < b$).

Indeed, since $\widetilde{X} = \lim U_i$, we get $\widetilde{Y} = \lim Y \times_X U_i$. But $Y \times_X U_i \to U_i$ are quasi-compact and quasi-separated, so $Y \times_X U_i$ are also quasi-compact and quasi-separated. By the previous lemma, $\widetilde{Y} \to U'$ factors through some $Y \times_X U_i$. □

Theorem 17. *Suppose $\pi : Y \to X$ is finite, $\overline{x} \to X$ is a geometric point, $x \in X$ is the image of $\overline{x}$. Then $(\pi_*\mathcal{F})_{\overline{x}} = \prod_{y\in\pi^{-1}(x)} \mathcal{F}_{\overline{y}}^{[k(y):k(x)]_{\text{sep}}}$.*

Proof. Denote $d(y) = [k(y) : k(x)]_{\text{sep}}$. Verify that $\widetilde{Y} = \sqcup_{y\in\pi^{-1}(x)} \sqcup_{d(y)} \operatorname{Spec} \mathscr{O}_{Y,\overline{y}}$. By the previous theorem, $(\pi_*\mathcal{F})_{\overline{x}} = \operatorname{colim}_{\widetilde{Y}\to U'\to Y} \mathcal{F}(U')$; also $\prod_{y\in\pi^{-1}(x)} \mathcal{F}_{\overline{y}}^{d(y)} = \operatorname{colim}_{(U_i):\ \{y_i\to U_i\to Y | y_i\in\widetilde{Y}\}} \prod_{y\in\pi^{-1}(x)} \prod_{d(y)} \mathcal{F}(U_i)$. Taking $U' = \sqcup U_i$, it suffices to show that $\{\widetilde{Y} \to U' \to Y$: images of the $\operatorname{Spec} \mathscr{O}_{Y,\overline{y}}$'s in $\widetilde{Y}$ lies in different connected components of $U'\}$ form a cofinal system in all $\{\widetilde{Y} \to U' \to Y\}$, which is obviously correct. □

Corollary 18. *Suppose $\pi : Y \to X$ is finite. Then $\pi_* : \mathsf{Ab}(X_{\text{ét}}) \to \mathsf{Ab}(Y_{\text{ét}})$ is exact.*

Corollary 19. *Suppose $j : U \to X$ is open immersion, $i : Z \to X$ is closed immersion, and $U \cup Z = X$ as sets. Then $0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$ is exact.*

Proof. Check on the stalks (note that closed embeddings are finite). □

Proposition 20. *Notation as the previous corollary. Let T be the category where objects are $\{(\mathcal{F}_1, \mathcal{F}_2, \phi) \mid \mathcal{F}_1 \in \mathsf{Ab}(Z_{\text{ét}}), \mathcal{F}_2 \in \mathsf{Ab}(U_{\text{ét}}), \phi : \mathcal{F}_1 \to i^*j_*\mathcal{F}_2\}$, and morphisms are pairs of morphisms $(\mathcal{F}_1 \to \mathcal{F}_1', \mathcal{F}_2 \to \mathcal{F}_2')$ that commute with ϕ. Then we have an equivalence of categories $\mathsf{Ab}(X_{\text{ét}}) \to T : \mathcal{F} \mapsto (i^*\mathcal{F}, j^*\mathcal{F}, i^*(\mathcal{F} \to j_*j^*\mathcal{F}))$.*

Proof. The essential inverse is given by $(\mathcal{F}_1, \mathcal{F}_2, \phi) \mapsto i_*\mathcal{F}_1 \times_{i_*i^*j_*\mathcal{F}_2} j_*\mathcal{F}_2$ (check yourself if you want). □

Corollary 21. $\mathsf{Ab}(X_{\text{ét}}) = \mathsf{Ab}((X_{\text{red}})_{\text{ét}})$.

5 Comparison of quasi-coherent cohomology and étale cohomology

Definition 22. Recall that $\mathsf{Ab}(X_{\text{ét}})$ is an abelian category with enough injective objects. For $U \in X_{\text{ét}}$, $\mathcal{F} \in \mathsf{Ab}(X_{\text{ét}})$, define the **étale cohomology** $H^p_{\text{ét}}(U, \mathcal{F}) := R^p\Gamma(U, -)(\mathcal{F})$.

Suppose X is a scheme, $\mathscr{F}$ is a quasi-coherent (Zariski) sheaf on X. Recall that we have defined $\mathscr{F}_{\text{ét}}(U) := \Gamma(U, (U \to X)^*\mathscr{F})$ and proved that $\mathscr{F}_{\text{ét}}$ is an étale sheaf. In this section, we will show that $H^p_{\text{ét}}(X, \mathscr{F}_{\text{ét}}) = H^p(X, \mathscr{F})$ for all p.

Lemma 23. *Suppose $\mathcal{I}$ is an injective object in $\mathsf{Ab}(X_*)$, where $* \in \{\text{ét}, \text{zar}\}$. Then $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for any covering $\mathcal{U} = (U_i \to U)$ and $p > 0$.*

Proof. The complex $\cdots \to \bigoplus_{i_0 < \cdots < i_n} (U_{i_0 \cdots i_n} \to U)_! \underline{\mathbb{Z}} \to \cdots$ is exact at all indexes $n > 0$ (check it on stalks). Apply $\mathrm{Hom}(-, \mathcal{I})$ to it, so $\check{C}^\bullet(\mathcal{U}, \mathcal{I})$ is exact at indexes $n > 0$, hence $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ for $p > 0$. □

Lemma 24. *Suppose $\mathcal{C} = X_*$, where $* \in \{\text{ét}, \text{zar}\}$, and $\mathcal{C}'$ is a site such that $\mathsf{Cat}(\mathcal{C}')$ is a full subcategory of $\mathsf{Cat}(\mathcal{C})$, with coverings in $\mathcal{C}'$ all being coverings in $\mathcal{C}$, satisfying*

(1) If $U' \to U \leftarrow U''$ are in $\mathcal{C}'$, then $U' \times_U U''$ exists in $\mathcal{C}$ and lies in $\mathcal{C}'$.

(2) For any covering $(U_i \to U)$ in $\mathcal{C}$, $U \in \mathcal{C}'$, there exists a refinement $(V_i \to U)$ a covering in $\mathcal{C}'$.

Assume that $\mathcal{F} \in \mathsf{Ab}(\mathcal{C})$, such that for any covering $\mathcal{U} = (U_i \to U)$ in $\mathcal{C}'$ and $p > 0$, $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$. Then for any $U \in \mathcal{C}'$, $p > 0$ we have $H^p_*(U, \mathcal{F}) = 0$.

Proof. Suppose $0 \to \mathcal{F} \to \mathcal{E} \xrightarrow{q} \mathcal{Q} \to 0$ is exact. We claim $\mathcal{E}(U) \to \mathcal{Q}(U)$ are surjective for all $U \in \mathcal{C}'$. Indeed, suppose $s \in \mathcal{Q}(U)$. Locally, s is in the image of $\mathcal{E}$. By condition (2), we may take a covering $\mathcal{U} = (V_i \to U)$ with each $V_i \in \mathcal{C}'$, such that $s|_{V_i} = q(t_i)$ for some $t_i \in \mathcal{E}(V_i)$. Let $u_{ij} = t_i - t_j \in \mathcal{F}(V_i \times_U V_j)$, so $(u_{ij}) \in \check{Z}^1(\mathcal{U}, \mathcal{F})$. Since $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, there are some $u_i \in \mathcal{F}(V_i)$ such that $u_{ij} = u_i - u_j$. Therefore $(t_i - u_i)$ glue to some $t \in \mathcal{E}(U)$, such that $q(t) = s$.

Therefore, for any covering $\mathcal{U} = (U_i \to U)$ in $\mathcal{C}'$, the sequence $0 \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{E}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$ is exact, hence we get the corresponding long exact sequence.

Take $\mathcal{E}$ injective. By the previous lemma, $\check{H}^p(\mathcal{U}, \mathcal{E}) = 0$, $\forall p > 0$. Now we use induction: (1) By $\mathcal{E}(U) \to \mathcal{Q}(U) \to H^1_*(U, \mathcal{F}) \to H^1_*(U, \mathcal{E}) = 0$, we get $H^1_*(U, \mathcal{F}) = 0$; (2) Since $H^p_*(U, \mathcal{F}) = H^{p-1}_*(U, \mathcal{Q})$ and $\mathcal{Q}$, the induction works. □

Lemma 25. *Suppose X is an affine scheme, and $\mathscr{F}$ is a quasi-coherent on X. Then $H^p_{\text{ét}}(X, \mathscr{F}_{\text{ét}}) = 0$ for all $p > 0$.*

Proof. In the previous lemma, take $\mathcal{C} = X_{\text{ét}}$, $\mathcal{C}' = \{\text{affine étale } U \to X\}$, with coverings $\{(U_i \to U)_I \in \text{Cov}(\mathcal{C}) \mid I \text{ finite}\}$, so conditions (1)(2) hold. Therefore it suffices to show that $\check{H}^p(\mathcal{U}, \mathscr{F}_{\text{ét}}) = 0$ for any covering $\mathcal{U} = (U_i \to U)_I$ in $\mathcal{C}'$. Since I is finite, we may take $V = \sqcup U_i \to U$. Calculating the Čech complexes, we get $\check{H}^p(\mathcal{U}, \mathscr{F}_{\text{ét}}) = \check{H}^p((V \to U), \mathscr{F}_{\text{ét}})$, so we reduce to the covering $V \to U$. Let $V = \operatorname{Spec} B$, $X = \operatorname{Spec} A$, so $A \to B$ is faithfully flat, and it suffices to show the Čech complex

$$\check{C}^{\bullet} : 0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M \to \cdots$$

is exact. This is a standard argument: (1) If $A \to B$ admits a section $s : B \to A$, then $(b_0, \cdots, b_n, m) \mapsto (b_1, \cdots, b_n, s(b_0)m)$ gives a homotopy $\text{id}_{\check{C}^{\bullet}} \simeq 0$, hence the homology vanishes. (2) In general, by faithfully flatness, it suffices to show $B \otimes_A \check{C}^{\bullet} : 0 \to M_B \to C \otimes_B M_B \to C \otimes_B C \otimes_B M_B \to \cdots$ is exact, where $C = B \otimes_A B$. This is already proved, since $B \to C$ admits a section $C = B \otimes_A B \to B : (b_1, b_2) \mapsto b_1 b_2$. $\square$

Theorem 26. *Suppose X is a scheme, $\mathscr{F}$ is a quasi-coherent sheaf on X. Then $H^p_{\text{ét}}(X, \mathscr{F}_{\text{ét}}) = H^p(X, \mathscr{F})$ for all p.*

Proof. We have commutative diagram

$$\begin{array}{ccc} \mathsf{Ab}(X_{\text{ét}}) & \xrightarrow{r} & \mathsf{Ab}(X_{\text{zar}}) \\ \downarrow{\scriptstyle v} & & \uparrow{\scriptstyle u = (\cdot)^{\#}} \\ \mathsf{PAb}(X_{\text{ét}}) & \xrightarrow{r'} & \mathsf{PAb}(X_{\text{zar}}) \end{array}$$

where r, r' are restrictions, v is the forgetful functor, and u is sheafification. Then r' and u are exact, so $R^p r(\mathcal{F}) = u \circ r' \circ R^p v(\mathcal{F}) = u \circ r'(U \mapsto H^p_{\text{ét}}(U, \mathcal{F})) = u \circ (U \mapsto H^p_{\text{ét}}(U, \mathcal{F})) = (U \mapsto H^p_{\text{ét}}(U, \mathcal{F}))^{\#,\text{zar}}$.

By the previous lemma, if $\mathscr{F}$ is quasi-coherent sheaf on X, then $H^p_{\text{ét}}(U, \mathscr{F}_{\text{ét}}) = 0$ for any U affine and $p > 0$. Hence $R^p r(\mathscr{F}_{\text{ét}}) = 0$ for $p > 0$.

Suppose $\mathcal{I}$ is an injective object in $\mathsf{Ab}(X_{\text{ét}})$. For any Zariski covering $\mathcal{U}$ of X, since $\check{C}^{\bullet}_{\text{zar}}(\mathcal{U}, r(\mathcal{I})) = \check{C}^{\bullet}_{\text{ét}}(\mathcal{U}, \mathcal{I})$, we get $\check{H}^p_{\text{zar}}(\mathcal{U}, r(\mathcal{I})) = \check{H}^p_{\text{ét}}(\mathcal{U}, \mathcal{I}) = 0$ for $p > 0$ (Lemma 23). Therefore, by Lemma 24, $r(\mathcal{I})$ is an acyclic object for the sheaf cohomology.

Now take an injective resolution $0 \to \mathscr{F}_{\text{ét}} \to \mathcal{I}^{\bullet}$. By definition, $h^p(r(\mathcal{I}^{\bullet})) = R^p r(\mathscr{F}_{\text{ét}}) = 0$ (as proved before), so $0 \to r(\mathscr{F}_{\text{ét}}) \to r(\mathcal{I}^{\bullet})$ is exact. Now let's use the acyclic objects $r(\mathcal{I}^{\bullet})$ to calculate the sheaf cohomology of $r(\mathscr{F}_{\text{ét}}) = \mathscr{F}$: $H^p(X, \mathscr{F}) = h^p(\Gamma(X, r(\mathcal{I}^{\bullet}))) = h^p(\Gamma_{\text{ét}}(X, \mathcal{I}^{\bullet})) = H^p_{\text{ét}}(X, \mathscr{F}_{\text{ét}})$. $\square$

Remark. In fact, by constructing an exact left adjoint for r, we can show $r(\mathcal{I})$ is injective.