Notes for Étale Cohomology Seminar 4

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Today we discuss more cohomological properties in $\mathsf{Ab}(X_{\text{ét}})$. The main reference is Chapter 9-11 in *Lectures on Etale Cohomology.* Most results can be viewed as an analogue of sheaf cohomology (see Section 3.2, 3.3, 3.4, 3.6 in [Hartshorne]).

1 Some homological algebra

Theorem 1 (Grothendieck spectral sequence)**.** *Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories, $\mathcal{A}, \mathcal{B}$ has enough injectives, $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ are left exact functors, and F sends injective objects to acyclic objects of G. Then $\forall X \in \mathcal{A}$, there is a spectral sequence such that $E_2^{p,q} = R^q G(R^p F X) \Rightarrow R^{p+q}(G \circ F)X$.*

Proposition 2 (Dimension axiom)**.** *Suppose $X = \operatorname{Spec} k^{\text{sep}}$. Then for all $\mathcal{F} \in \mathsf{Ab}(X_{\text{ét}})$ we have $H^p(X_{\text{ét}}, \mathcal{F}) = 0$, $\forall p > 0$.*

Proof. In this case, $H^p(X_{\text{ét}}, -) = \mathcal{F}_{\text{pt}}$ is exact. □

Definition 3. Since $\operatorname{Hom}_{X_{\text{ét}}}(\mathcal{F}, -) : \mathsf{Ab}(X_{\text{ét}}) \to \mathsf{Ab}$ and $\mathscr{H}\!om_{X_{\text{ét}}}(\mathcal{F}, -) : \mathsf{Ab}(X_{\text{ét}}) \to \mathsf{Ab}(X_{\text{ét}})$ are left exact, we can define $\operatorname{Ext}^p_{X_{\text{ét}}}(\mathcal{F}, \mathcal{G}) := (R^p \operatorname{Hom}_{X_{\text{ét}}}(\mathcal{F}, -))(\mathcal{G})$ and $\mathscr{E}\!xt^p_{X_{\text{ét}}}(\mathcal{F}, \mathcal{G}) := (R^p \mathscr{H}\!om_{X_{\text{ét}}}(\mathcal{F}, -))(\mathcal{G})$.

Remark. Generally there aren't enough projectives in $\mathsf{Ab}(X_{\text{ét}})$, so we can't define it by $(L_p \operatorname{Hom}_{X_{\text{ét}}}(-, \mathcal{G}))(\mathcal{F})$.

Proposition 4. *For short exact sequences $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$, $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have long exact sequences*

$$\cdots \to \operatorname{Ext}^p_{X_{\text{ét}}}(\mathcal{F}, \mathcal{G}') \to \operatorname{Ext}^p_{X_{\text{ét}}}(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^p_{X_{\text{ét}}}(\mathcal{F}, \mathcal{G}'') \to \operatorname{Ext}^{p+1}_{X_{\text{ét}}}(\mathcal{F}, \mathcal{G}') \to \cdots,$$

$$\cdots \to \operatorname{Ext}^p_{X_{\text{ét}}}(\mathcal{F}'', \mathcal{G}) \to \operatorname{Ext}^p_{X_{\text{ét}}}(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^p_{X_{\text{ét}}}(\mathcal{F}', \mathcal{G}) \to \operatorname{Ext}^{p+1}_{X_{\text{ét}}}(\mathcal{F}'', \mathcal{G}) \to \cdots.$$

Proof. The first exact sequence follows from the general theory of derived functors. For the second, suppose $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ is injective resolution. Then $0 \to \operatorname{Hom}(\mathcal{F}'', \mathcal{I}^\bullet) \to \operatorname{Hom}(\mathcal{F}, \mathcal{I}^\bullet) \to \operatorname{Hom}(\mathcal{F}', \mathcal{I}^\bullet) \to 0$ is an exact sequence of complexes, so we get the long exact sequence. □

2 Cohomology with support

Definition 5. Suppose $i : Z \hookrightarrow X$ is closed subscheme, and $j : U = X\backslash Z \hookrightarrow X$. The **sections with support on** Z is $\Gamma_Z(X, \mathcal{F}) := \ker(\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F}))$.

Remark. It is analogue of $H^p(X, Z; G)$ in the cohomology of topological spaces.

Recall that we have an exact sequence

$$0 \to j_! j^* \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to i_* i^* \underline{\mathbb{Z}} \to 0.$$

Observe that $\mathrm{Hom}(\underline{\mathbb{Z}}, \mathcal{F}) = \mathcal{F}(X)$ and $\mathrm{Hom}(j_! j^* \underline{\mathbb{Z}}, \mathcal{F}) = \mathrm{Hom}(j^* \underline{\mathbb{Z}}, j^* \mathcal{F}) = \mathrm{Hom}(\underline{\mathbb{Z}}, j^* \mathcal{F}) = (j_* \mathcal{F})(X) = \mathcal{F}(U)$. Therefore we get a left exact sequence $0 \to \mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, \mathcal{F}) \to \mathcal{F}(X) \to \mathcal{F}(U)$. Hence

Proposition 6. *(1)* $\mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, \mathcal{F}) = \Gamma_Z(X, \mathcal{F})$*.*

(2) If we define the derived functor

$$H^p_Z(X, \mathcal{F}) := (R^p \Gamma_Z(X, -))(\mathcal{F}) = \mathrm{Ext}^p(i_* i^* \underline{Z}, \mathcal{F})$$

then there is a long exact sequence

$$\cdots \to H^p_Z(X, \mathcal{F}) \to H^p(X, \mathcal{F}) \to H^p(U, \mathcal{F}) \to H^{p+1}_Z(X, \mathcal{F}) \to \cdots .$$

Theorem 7 (Excision)**.** *Suppose $\pi : X' \to X$ is étale, $Z \subset X$ is closed subset, $Z' = \pi^{-1}(Z)$, such that $\pi|_{Z'} : Z' \to Z$ is an isomorphism. Then the canonical map $H^p_Z(X_{\text{ét}}, \mathcal{F}) \to H^p_{Z'}(X'_{\text{ét}}, \mathcal{F})$ is an isomorphism.*

Proof. Define $U = X\backslash Z$, $U' = X'\backslash Z'$. The map $\Gamma_Z(X, \mathcal{F}) \to \Gamma_{Z'}(X', \mathcal{F}|_{X'})$ is given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma_Z(X, \mathcal{F}) & \longrightarrow & \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma_Z(X', \mathcal{F}) & \longrightarrow & \Gamma(X', \mathcal{F}) & \longrightarrow & \Gamma(U', \mathcal{F}) \end{array}$$

We claim that this is an isomorphism.

(1) Suppose $s \in \Gamma_Z(X, \mathcal{F})$ maps to 0. Then $s \in \Gamma(X, \mathcal{F})$ maps to 0 in $\Gamma(U, \mathcal{F})$ and $\Gamma(X', \mathcal{F})$. But $(U \to X, X' \to X)$ is a covering of X, so $s = 0$.

(2) Suppose we have $s' \in \Gamma_{Z'}(X', \mathcal{F}|_{X'}) \subset \Gamma(X', \mathcal{F})$, so s' maps to $0 \in \Gamma(U', \mathcal{F})$. Consider $(s', 0) \in \Gamma(X', \mathcal{F}) \times \Gamma(U, \mathcal{F})$. In order to glue we need to verify

(a) $0|_{U \times_X U \xrightarrow{\mathrm{pr}_1} U} = 0|_{U \times_X U \xrightarrow{\mathrm{pr}_2} U}$.

(b) $s'|_{U \times_X X'} = 0|_{U \times_X X'}$.

(c) $s|_{X' \times_X X' \xrightarrow{\mathrm{pr}_1} X'} = s|_{X' \times_X X' \xrightarrow{\mathrm{pr}_2} X'}$.

The first is trivial. The second is because $U \times_X X' = U'$, so both sides are zero. The third is because $X' \times_X X' = (Z' \times_X Z') \cup (U' \times_X U')$, $Z' \cong Z$, and on $U' \times_X U'$ both sides are zero.

Therefore we have proved this is true for $p = 0$. For the general case, since π^* is exact, it suffices to show that π^* preserves injectives. In fact, the extension by zero functor can be defined on an étale map: suppose $\mathcal{G} \in \mathsf{Ab}(X'_{\text{ét}})$, then $\pi_! \mathcal{G}$ is the sheafification $(U \mapsto \bigoplus_{f \in \mathrm{Hom}_X(U,X')} \mathcal{G}(U))^{\#}$. Then $(\pi_!, \pi^*)$ is an adjoint pair, and $\pi_!$ is exact, so π^* preserves injectives (see last week's discussion on extension by zero functors). $\square$

Proposition 8 (Mayer Vietoris sequence)**.** *Suppose $X = U \cup V$, U, V are open subschemes. Then we have a long exact sequence $\cdots \to H^p(X, \mathcal{F}) \to H^p(U, \mathcal{F}) \oplus H^p(V, \mathcal{F}) \to H^p(U \cap V, \mathcal{F}) \to H^{p+1}(X, \mathcal{F}) \to \cdots$ is exact.*

Proof. It is the Ext long exact sequence for $0 \to (U \cap V \to X)_! \underline{\mathbb{Z}} \to (U \to X)_! \underline{\mathbb{Z}} \oplus (V \to X)_! \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to 0$. $\square$

3 Čech cohomology

We are going to discuss Čech cohomology under a more general case, i.e. in $\mathsf{PAb}(\mathcal{C})$ or $\mathsf{Ab}(\mathcal{C})$ for $\mathcal{C}$ a site. We will need the following theorem (the case for $\mathsf{Ab}(X_{\text{ét}})$ was proved in the last time).

Theorem 9. *Suppose $\mathcal{C}$ is a site. Then*

(1) $\mathsf{PAb}(\mathcal{C})$ is abelian, where limits and colimits exist and are calculated sectionwisely ([Stacks] Tag `03A6`).

(2) $\mathsf{PAb}(\mathcal{C})$ has enough injective objects ([Stacks] Tag `01DJ`).

(3) $\mathsf{Ab}(\mathcal{C})$ is an abelian category, where limits and colimits exist, and filtered colimits are exact ([Stacks] Tag `03CM`)

(4) $\mathsf{Ab}(\mathcal{C})$ has enough injective objects ([Stacks] Tag `01DP`).

Definition 10. Suppose $\mathcal{C}$ is a site, $\mathcal{F} \in \mathsf{PAb}(\mathcal{C})$, $\mathcal{U} = (U_i \to U)_I \in \mathrm{Cov}(\mathcal{C})$. Denote

$$U_{i_0 \cdots i_p} := U_{i_0} \times_U \cdots \times_U U_{i_p}.$$

Define the **Čech complex**

$$\check{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{(i_0, \cdots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0 \cdots i_p}),$$

$$d^p : (s_{i_0 \cdots i_p}) \mapsto (\sum_{k=0}^{p+1} (-1)^k \operatorname{pr}^*_{i_0 \cdots \hat{i_k} \cdots i_{p+1}} s_{i_0 \cdots \hat{i_k} \cdots i_{p+1}}),$$

and let $\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p(\check{C}^\bullet(\mathcal{U}, \mathcal{F}))$ be the **Čech cohomology**.

Remark. In Zariski topology, for any $U_i \subset U$ we have $U_i \times_U U_i = U_i$, so the two projections coincide. In a general site, the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : U_i \times_U U_i \to U_i$ may be different. In order for a section $s_i \in \mathcal{F}(U_i)$ to glue to some $s \in \mathcal{F}(U)$, a necessary condition is $\mathrm{pr}_1^* s = \mathrm{pr}_2^* s$. Therefore, unlike the Zariski case, when defining Čech complexes on general sites, we need to include those $U_{i_0 \cdots i_p}$ with repetitive indexes $i_a = i_b \in I$.

Remark. Take

$$\mathbb{Z}_{\mathcal{U},p} : V \mapsto \bigoplus_{(i_0,\cdots,i_p)\in I^{p+1}, f:V\to U_{i_0\cdots i_p}} \mathbb{Z} \in \mathsf{PAb}(\mathcal{C}).$$

Then $\mathbb{Z}_{\mathcal{U},\bullet}$ is a complex of presheaves, which is exact in deg $\neq 0$ (See [Stacks] Tag `03AT`). Clearly $\mathrm{Hom}_{\mathsf{PAb}}(\mathbb{Z}_{U,\bullet}, \mathcal{F}) = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$. In particular,

Proposition 11. *If $\mathcal{F}$ is injective in $\mathsf{PAb}(\mathcal{C})$, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.*

Note that any injective sheaf is also injective as a presheaf (reason: the forgetful functor $u : \mathsf{Ab}(\mathcal{C}) \to \mathsf{PAb}(\mathcal{C})$ has a left adjoint, the sheafification, which is left exact, so u preserves injective objects). Therefore,

Proposition 12. *If $\mathcal{F}$ is injective in $\mathsf{Ab}(\mathcal{C})$, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.*

Suppose $\mathcal{F} \in \mathsf{PAb}(\mathcal{C})$. Let's consider the following spectral sequence. Take an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in PAb. Take $E^{pq} = \check{C}^q(\mathcal{U}, \mathcal{I}^p)$. In the right orientation,

$$E_1^{pq} = \begin{cases} \check{C}^q(\mathcal{U}, \mathcal{F}), & p = 0, \\ 0, & p \neq 0, \end{cases}$$

$$E_2^{0q} = \check{H}^q(\mathcal{U}, \mathcal{F}).$$

In the up orientation, (by the proposition above)

$$E_1^{pq} = \begin{cases} \check{H}^0(\mathcal{U}, \mathcal{I}^p), & q = 0, \\ 0, & q \neq 0, \end{cases}$$

$$E_2^{p0} = (R^p \check{H}^0(\mathcal{U}, -))(\mathcal{F}).$$

Therefore,

Proposition 13. *In the category $\mathsf{PAb}(\mathcal{C})$, $\check{H}^p(\mathcal{U}, \mathcal{F}) = (R^p \check{H}^0(\mathcal{U}, -))(\mathcal{F})$.*

Now we limit to the category of sheaves $\mathsf{Ab}(\mathcal{C})$.

Let $u : \mathsf{Ab}(\mathcal{C}) \to \mathsf{PAb}(\mathcal{C})$ be the forgetful functor. Clearly $R^q u(\mathcal{F})$ is the presheaf $V \mapsto H^q(V, \mathcal{F})$. Together with the proposition above, the Grothendieck spectral sequence of

$$\mathsf{Ab}(\mathcal{C}) \xrightarrow{u} \mathsf{PAb}(\mathcal{C}) \xrightarrow{\check{H}^0} \mathsf{Ab}.$$

gives $E_2^{pq} = \check{H}^q(\mathcal{U}, V \mapsto H^p(V, \mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F})$ (i.e. the **Čech-to-derived functor spectral sequence**). In particular, take $p = 0$, so there are natural maps

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{F}).$$

(Alternatively, you can take an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$, and use a similar procedure as the presheaf case to obtain these maps.)

Definition 14. $\mathcal{V} = (V_j \to X)_J$ is called a **refinement** of $\mathcal{U} = (U_i \to X)_I$ if there is some $\alpha : J \to I$ such that $V_j \to X$ factors through $U_{\alpha(j)}$. Then we get a natural chain map $\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{V}, \mathcal{F})$ (that depends on $\alpha : J \to I$).

For any two refinements $\alpha, \beta : J \to I$, their induced chain maps $\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{V}, \mathcal{F})$ are homotopic (hint: something like $h((s_{i_0 \cdots i_p})) = (\sum_{k=0}^{p-1} (-1)^k s_{\alpha(j_0) \cdots \alpha(j_k) \beta(j_k) \cdots \beta_{j_p}})$ gives a homotopy). Therefore the induced maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$ don't depend on the choice of map $J \to I$.

Definition 15. $\check{H}^p(X, \mathcal{F}) := \operatorname{colim}_{\mathcal{U} \text{ covering}} \check{H}^p(\mathcal{U}, \mathcal{F})$. Therefore there are natural maps $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$.

Theorem 16 (Theorem 10.2 in book)**.** *Suppose X is Noetherian, such that any finite subset of X is contained in an affine open (for example if X is quasi-projective over k). Then the natural maps $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ given previously are isomorphisms, for $\mathcal{F} \in \mathsf{Ab}(X_{\text{ét}})$.*

Theorem 17 (Theorem 10.9 in book)**.** *Suppose I is directed set, $(X_i)_{i \in I}$ is an inverse system of X-schemes, such that X_i are quasi-compact, with affine transition maps. Therefore $X_\infty = \lim X_i$ exists. For any sheaf $\mathcal{F}$ on $X_{\text{ét}}$ we have isomorphisms*

$$\operatorname{colim}_i H^p(X_{i,\text{ét}}, \mathcal{F}_i) \xrightarrow{\sim} H^p(X_{\infty,\text{ét}}, \mathcal{F}_\infty),$$

where the sheaves on X_i, X_∞ are given by pullback.

3.1 Principal homogeneous spaces

Definition 18. Suppose $\mathcal{C}$ is a site, $\mathcal{G}$ is a sheaf of groups (possibly non-abelian). Let $\mathcal{S}$ be a sheaf of sets, and $\mathcal{G}$ acts on $\mathcal{S}$. $\mathcal{S}$ is called a **principal homogeneous space** (or **torsor**) for $\mathcal{G}$ if

(a) For any $U \in \mathcal{C}$ there is a covering $(U_i \to U)$ such that $\mathcal{S}(U_i)$ is non-empty for all i.

(b) For any $U \in \mathcal{C}, s \in \mathcal{S}(U)$, the map $\mathcal{G}(U) \to \mathcal{S}(U) : g \mapsto gs$ is a bijection (this condition is equivalent to $\mathcal{G} \times \mathcal{S} \to \mathcal{S} \times \mathcal{S} : (g, s) \mapsto (gs, s)$ is an isomorphism).

Definition 19. Suppose $\mathcal{C}$ has a final object U, and $\mathcal{U} = (U_i \to U)$ is a covering. $\mathcal{S}$ is called **split by** $\mathcal{U}$ if $\mathcal{S}(U_i)$ are non-empty for all i.

If the $\mathcal{G}$-torsor $\mathcal{S}$ is split by $\mathcal{U}$, then we have some "cocycle condition". Indeed, for $\{s_i \in \mathcal{S}(U_i)\}$, take (unique) element $g_{ij} \in \mathcal{G}(U_{ij})$ such that $g_{ij}s_i|_{U_{ij}} = s_j|_{U_{ij}}$, so

$$g_{ik}s_i|_{U_{ijk}} = s_k|_{U_{ijk}} = g_{jk}s_j|_{U_{ijk}} = g_{jk}g_{ij}s_i|_{U_{ijk}},$$

hence $g_{ik}|_{U_{ijk}} = g_{jk}g_{ij}|_{U_{ijk}}$ by uniqueness.

Suppose there are some other $\{s'_i \in \mathcal{S}(U_i)\}$, so we can take $h_i \in \mathcal{G}(U_i)$ such that $s'_i = h_i s_i$, hence $g'_{ij}h_i|_{U_{ij}} = h_j|_{U_ij}g_{ij}$.

Therefore, if we define

$$\check{H}^1(\mathcal{U}, \mathcal{G}) := \{(g_{ij}) \in \prod_{i,j} \Gamma(U_{ij}, \mathcal{G}) \mid g_{ik}|_{U_{ijk}} = g_{jk}g_{ij}|_{U_{ijk}}\}/\sim,$$

where "$\sim$" is given by

$$(g'_{ij}) \sim (g_{ij}) \iff \text{there are } h_i \in \mathcal{G}(U_i) \text{ such that} g'_{ij} = h_j g_{ij} h_i^{-1},$$

then

Proposition 20. *There is a natural bijection between the set $\check{H}^1(\mathcal{U}, \mathcal{G})$ and isomorphism classes of $\mathcal{G}$-torsors splitting by $\mathcal{U}$.*

Suppose $\mathcal{G} \in \mathsf{Ab}(\mathcal{C})$. In this case, our definition for $\check{H}^1(\mathcal{U}, \mathcal{G})$ here coincides with the previous one by Čech cohomology. Therefore, $\check{H}^1(U, \mathcal{G}) = \operatorname{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{G})$ is the set of all $\mathcal{G}$-torsors.

3.2 Comparison for $p = 1$

Similar to the Zariski case, for $\mathcal{F} \in \mathsf{Ab}(\mathcal{C})$, in general $\check{H}^p(X, \mathcal{F}) \neq H^p(X, \mathcal{F})$, but they are equal when $p = 1$.

Theorem 21. *Suppose $\mathcal{F} \in \mathsf{Ab}(\mathcal{C})$. Then we have natural isomorphisms $\check{H}^1(X, \mathcal{F}) \xrightarrow{\sim} H^1(X, \mathcal{F})$.*

Proof. Let $\mathcal{H} = \mathcal{H}^1_{\text{pre}}(X, \mathcal{F})$ be the presheaf $U \mapsto H^1(U, \mathcal{F})$. Embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$, and let $\mathcal{Q}$ be the quotient, so $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ is exact.

Claim: $\operatorname{colim}_{\mathcal{U}} \prod_i \mathcal{H}(U_i) = 0$. Indeed, suppose we have $(a_i)_i \in \prod_i \mathcal{H}(U_i)$. Note that $0 \to \mathcal{F}(U_i) \to \mathcal{I}(U_i) \to \mathcal{Q}(U_i) \to \mathcal{H}(U_i) \to H^1(U_i, \mathcal{I}) = 0$ is exact. Let $b_i \in \mathcal{Q}(U_i)$ be some preimage of a_i. Since $\mathcal{I} \to \mathcal{Q}$ is locally surjective on sections, there is an open cover V_{ij} of U_i such that $b_i|_{V_{ij}} \in \operatorname{im}(\mathcal{I}(V_{ij}) \to \mathcal{Q}(V_{ij})) = \ker(\mathcal{Q}(V_{ij}) \to \mathcal{H}(V_{ij}))$, hence $a_i|_{V_{ij}} = 0$. Let $\mathcal{V} = \{V_{ij}\}$, which is clearly a refinement of $\mathcal{U}$, so the image of $(a_i)_i$ in $\prod_{i,j} \mathcal{H}(V_{ij})$ is 0. Hence the claim follows.

Since the presheaf $\mathcal{H}^1_{\text{pre}}(X,\mathcal{I}) : U \mapsto H^1(U,\mathcal{I})$ is 0, we get an exact sequence of chain complexes $0 \to \check{C}^\bullet(\mathcal{U},\mathcal{F}) \to \check{C}^\bullet(\mathcal{U},\mathcal{I}) \to \check{C}^\bullet(\mathcal{U},\mathcal{Q}) \to \check{C}^\bullet(\mathcal{U},\mathcal{H}) \to 0$. Let $D^\bullet(\mathcal{U}) = \operatorname{coker}(\check{C}^\bullet(\mathcal{U},\mathcal{F}) \to \check{C}^\bullet(\mathcal{U},\mathcal{I}))$, so there are exact sequences of chain complexes

$$0 \to \check{C}^\bullet(\mathcal{U},\mathcal{F}) \to \check{C}^\bullet(\mathcal{U},\mathcal{I}) \to D^\bullet(\mathcal{U}) \to 0,$$

$$0 \to D^\bullet(\mathcal{U}) \to \check{C}^\bullet(\mathcal{U},\mathcal{Q}) \to \check{C}^\bullet(\mathcal{U},\mathcal{H}) \to 0.$$

Clearly $\check{H}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(X)$, $\check{H}^0(\mathcal{U},\mathcal{I}) = \mathcal{I}(X)$, $\check{H}^0(\mathcal{U},\mathcal{Q}) = \mathcal{Q}(X)$. Also $\check{H}^1(\mathcal{U},\mathcal{I}) = 0$ (Proposition 12). Hence the long exact sequences of the two chain complexes above are

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to h^0(D^\bullet(\mathcal{U})) \to \check{H}^1(\mathcal{U},\mathcal{F}) \to 0 \to \cdots,$$

$$0 \to h^0(D^\bullet(\mathcal{U})) \to \mathcal{Q}(X) \to h^0(\check{C}^\bullet(\mathcal{U},\mathcal{H})) \to \cdots.$$

By definition, $h^0(\check{C}^\bullet(\mathcal{U},\mathcal{H}))$ is a quotient of $\check{C}^0(\mathcal{U},\mathcal{H}) = \prod_i \mathcal{H}(U_i)$. With our claim above, we get $\operatorname{colim}_{\mathcal{U}} h^0(\check{C}^\bullet(\mathcal{U},\mathcal{H})) = 0$. Therefore $\operatorname{colim}_{\mathcal{U}} h^0(D^\bullet(\mathcal{U})) = \mathcal{Q}(X)$. Then take colimit in the first sequence, so

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{Q}(X) \to \operatorname{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U},\mathcal{F}) \to 0.$$

On the other hand, $0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{Q}(X) \to H^1(X,\mathcal{F}) \to 0$ is exact. Therefore $\operatorname{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U},\mathcal{F}) = H^1(X,\mathcal{F})$. □